Method to the Madness: A Game Theoretical Analysis of the USA and North Korea's Standoff

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In the past year the world has been threatened with the prospect of nuclear war, by the seemingly impulsive behavior of the North Korean regime. In this paper, Marcel Jaensch opens the possibility that their behavior is in fact rational and is a key part of their survival strategy. Marcel uses game theory to display the value of uncertainty around their capabilities to strike the US and its allies, showing that the US will only attack if they can be almost certain of an inability to strike back. This insight gives rationale to the North's seemingly erratic behavior and strategy, showing that it may, from the regimes point of view, be optimal.

Intoduction

In the last months, the North Korean military launched numerous missile tests, leaving the world trembling, causing a rush to safe haven assets and giving rise to semi-humorous rhetoric by the US president. The strategy of North Korea is to develop an Intercontinental Ballistic Missile (ICBM) that is capable both in range and accuracy to reach a major American city. On November 28th, North Korea claims to have successfully tested for the first time such an ICBM, which however cannot be verified (Kong, 2017). Therefore, as the DPRK continues their missile tests, it is instructive to use game theory to analyse the options of the North Koreans and the resulting response by the United States. The US intelligence community will be fully aware of such further tests. However, the outcome of development is uncertain to the United States. This information asymmetry sets the stage for a game theoretical analysis of the interactions between both players. The paper will start by outlining the model, its assumptions and the payoffs before representing it in diagrammatic form. The essay will the outline the equilibria of the game and analyse their significance, before discussing limitations and possible extensions to the model. The main insight of this paper is the absence of signalling by North Korea, and the fact that uncertainty is key to the regime's preservation. This value of uncertainty to the regime indicates that their seemingly reckless and impulsive behaviour is in fact calculated and strategic, and ultimately necessary for its survival.

Model

This paper models an extensive game with imperfect information involving the Democratic People's Republic of Korea (DPRK) and the United States as single players. The model represents a scenario in which the DPRK tests another Intercontinental Ballistic Missile. The outcome of the test will determine whether North Korea possess capabilities of attacking the United States mainland with nuclear missiles. Hence, a successful test will render the DPRK an ICBM capable type, and an unsuccessful test will make the DPRK an ICBM incapable type. The United States will be aware of such a test and will either play the same game with one of two probability distribution. If the US intelligence community deems the test as a failure, then nature will choose an ICBM success with probability 0.1 and an ICBM failure with probability 0.9. If, however, the US intelligence agencies categorise the test as a success, nature deems the ICBM test a success with probability 0.9 and a failure with probability 0.1.

Once the appropriate game has been selected given the intelligence analysis, nature moves first and determines the outcome of the test given the underlining probabilities. After nature's move, only the North Korean leadership themselves know the exact outcome of whether the test and, thus, the development of an ICBM was successful. The United States will only know the underlining discrete probability distribution and will never have complete information on whether North Korea possess ICBM capabilities. The United States will, however, form believes on the outcomes of the tests after observing North Korea's first move. Once the DPRK knows which type they are, they will choose whether to announce the development of such a threating weapon as a success or publicly claim it as a failure. After the announcement, the United States will react by either retreating and removing its military forces from the region, or by attacking North Korea. Hence, either the game will end in peace given the US retreats or in war, which might be a nuclear war leading to total annihilation. This assumption is made to simplify the analysis.

Assumptions

The first assumption is that the international community and the United States will be aware of such a missile launch. This is credible as North Korea most likely will issue a notice to airmen, known as a NOTAM. Such a warning will be most likely used by the North Korean's to notify the international community to avoid risking starting a war. Even if no such warning is issued, the US military using space-based sensors in conjunction with ground-based radar can detect a missile launch, its type, bearing, range and lastly whether it failed or succeeded. However, sometimes information is lacking, and occasionally conflicting, or wrong. Therefore, even though the international community is aware of a launch and can predict to a high certainty whether it was successful or not, a small error term persists (Hanham, 2017). This error term is the basis for the second assumption, the two sets of discrete probabilities. If the US intelligence community deems a test a failure, the error term is incorporated in the game by the fact that the ICBM test is a success with a probability of 0.1. If the test is deemed successful by the intelligence agencies, there is still a probability of 0.1 that the ICBM test was a failure.

The third assumption is that North Korea will publicly announce the outcome of the tests, even when it failed. While this might not be reasonable, it adds a layer of complexity that creates a model which yields an interesting equilibria. Fourth, it is assumed that the United States can only either retreat or attack. This simplifies the game tree by reducing the number of branches, effectively ruling out the status quo whereby the United States takes no action.

Lastly, the payoffs, in the model also rely on a set of assumptions about the actions and preferences of each player. However, it is assumed that the payoffs to the DPRK do not depend on the outcome of the ICBM test only on the actions of the United States. The DPRK would prefer the United States to retreat and remove its forces from the region rather than being attacked. An attack by the United States would end in a complete loss with or without ICBM capabilities. As for the United States, it is assumed that it has two sets of preferences depending on the outcome of the ICBM test. If the test is a success, the US would prefer to retreat rather than attack and vice versa if the ICBM test is a failure.

Overall, the United States would prefer to attack North Korea when it does not possess ICBM capabilities, as it can win the war without incurring destruction at home. Then, the United States second most favoured outcome is to retreat when the ICBM test was a success, as it ensures that major US cities do not get attacked. However, the United States would prefer less to retreat when the

ICBM test is a failure as it would have seceded without a credible threat. Lastly, the United States would least prefer to start a nuclear war with a North Korea that has ICBM capabilities, as this would lead to complete destruction of the US homeland. The range of values attached to the payoffs for both parties is between zero and five. These assumptions are represented in the table below.

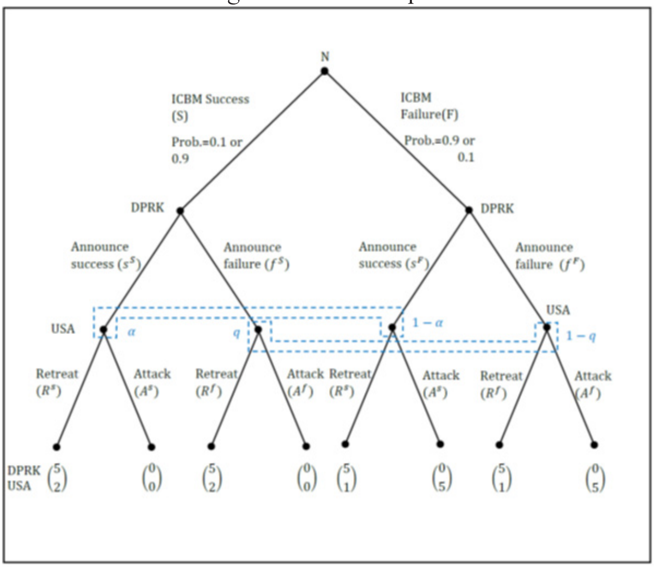
Table 1: Payoffs of the United States and DPRK in respect to outcomes

Outcome	USA Payoffs	DPRK Payoffs
Attack when ICBM test is a failure	5	0
Retreat when ICBM test is a success	2	5
Retreat when ICBM test is a failure	1	5
Attack when ICBM test is a success	0	0

Representation

The extensive form games with incomplete information described above are represented in the diagram below. As outlined earlier, the only difference between the two games are the two sets of underlining probabilities at the start of each game.

Figure 1: Extensive form games of nuclear politics between USA and DPRK



Equilibria

The first extensive Bayesian game, in which the US intelligence reports deem the ICBM test to have failed, yields two pure-strategy pooling equilibria. The first Perfect Bayesian Equilibrium (PBE) states that the DPRK will announce a successful ICBM test regardless of the actual outcome. The United States will in turn attack in both cases. This rest on parameters that the probability with which the USA believes the ICBM test was successful when the DPRK announces a success (a=Prob(S|s))is equal to the prior probability that the ICBM test was a success (α=0.1). It further rests on the assumption that the probability with which the USA believes the ICBM development was successful when the DPRK announces a failure (q=Prob(S|f)) is less than or equal to 0.67 (q$[0,{}^2/_3]$). The second Perfect Bayesian Equilibrium sets forth that the DPRK will announce a failure of the test irrespective of actual outcome and the United States will respond by attacking North Korea, given that q=0.1 and $\alpha[0, {}^2/_3]$. These sets of PBE are summarized below,

Figure 2: Set of PBE of Game 1

$$PBE = \left\{(s^S s^F, A^s A^f), \alpha = 0.1, q) : q \in \left[0, \frac{2}{3}\right]\right\} \cup \left\{(f^S f^F, A^s A^f), \alpha, q = 0.1) : \alpha \in \left[0, \frac{2}{3}\right]\right\}$$

In the second extensive Bayesian game, when the US intelligence community deems the ICBM test to be successful, there are four pure-strategy pooling equilibria. In the set of the first two Perfect Bayesian Equilibria, the DPRK announces that the ICBM tests were successful regardless of its validity. In both equilibria, given α=0.9 the United States will retreat when observing an announcement of success. However, in one equilibrium the USA will retreat even when a failure is announced if $q\in[{}^2/_3,1]$ and in the other equilibrium the USA will attack when failure is announced if $q\in[0,{}^2/_3]$. If $q={}^2/_3$, the United States is indifferent between retreating or attacking when failure is announced. In the second set of the other two Perfect Bayesian Equilibria, the DPRK announces that the ICBM tests were a failure regardless of actual events. In both equilibria, given q=0.9 the United States will retreat when observing an announcement of failure. However, in one equilibrium, the USA will retreat even when a success is announced if $\alpha\in[{}^2/_3,1]$ and in the other equilibrium the United States will attack when success is announced if $\alpha\in[0,{}^2/_3]$. If $\alpha={}^2/_3$, the United States is indifferent between retreating or attacking when success is announced. See the set of above described PBE on the next page (Figure 3: Set of PBE of Game 2)

$$U_{USA}(s^S s^F, R^f) = q(2) + (1-q)(1) = 1 + q$$

$$U_{USA}(s^S s^F, A^f) = q(0) + (1-q)(5) = 5 - 5q$$

$$U_{USA}(s^S s^F, R^f) \leq U_{USA}(s^S s^F, A^f) \text{ if and only if (iff)}$$

$$1 + q \leq 5 - 5q$$

$$6q \leq 4$$

$$q \leq \frac{2}{3}$$

Analysis and Extensions

The interesting fact about both games is that they yield exclusively pooling equilibria. In these equilibria, the sender, the DPRK, who observes the outcome of the ICBM tests chooses the same action, so that the sender's action gives the receiver, the United States, no information about the sender's type. Hence, the DPRK employs a pooling strategy to conceal whether it is an ICBM capable type or not. Intuitively, it is advantageous for the DPRK not to signal its type to the USA, as it would be inviting the US to attack in case it is incapable of launching nuclear ICBMs against the US mainland. Given that no signalling occurs, the United States uses the underlining probabilities of each game to form beliefs and best responses. As the one of underlining probabilities in each of the games is closest to one, the model approaches an extensive game of complete information. Hence, when it is extremely likely that the ICBM test failed, as in Game 1, the US will want to attack North Korea, which is the case in all PBE of Game 1. On the other hand, if it is very likely that North Korea possess ICBM capabilities, the US is better off retreating, as is evident in most of the PBE of Game 2. Hence, given the assumptions the predictions are realistic.

Nonetheless, the prior assumptions need to be assessed by their effect on the games' outcomes and how realistic they are. The first two assumptions, about the observance of a missile launch and the error term of an intelligence analysis, are both very much rooted in real-life scenarios as outlined above. The assumption that North Korea publicly announces a missile test failure is somewhat unrealistic, but adds a layer of complexity to the game.

However, the insightful pooling equilibrium is reached by assuming that the DPRK's payoff is equivalent when they announce success or failure of the ICBM, and only depends on the United States response. As the information transmission is costless and the interest between the DPRK and the US are not aligned, it is always more advantageous for North Korea to conceal their actual capabilities.

In the game theory literature, this is referred to as 'Cheap talk', first outlined in its basic form by Crawford and Sobel (1982). This type of game can be applied to any interaction in which an informed player, who is biased, advises a decision maker, where communication is direct, costless, non-binding and unverifiable. The standard example is that of a lobbyist informing a politician of the state of the industry they are representing (Munoz-Garcia and Toro-Gonzalez, 2016). Lastly, to limit the response of the USA to two options of either attack or retreat is too strong an assumption to make. It leaves out a third option of the US standing put, which would represent the status quo.

To model the game closer to reality, it can be amended by two aspects. First, the United States should be given the additional third option to stand firm, which should represent the status quo. The payoff for the status quo will have to be equal for both players. For the US, it has to take up a value which is greater than the payoff from retreating when the ICBM test is a success, but less than the payoff from attacking when the ICBM test is a failure (status quo $\in (2,5)$). Different from the original game, this third option would make retreating less reasonable for the US as the payoff from staying put is always higher. Additionally, to introduce signalling in this game it must be costly for North Korea to make the false announcement of a successful ICBM launch (s^F). This makes the game more realistic as it is extremely costly for the DPRK to maintain a façade of being ICBM capable. However, there will only be a separating equilibrium, indicative of signalling, if the costs of lying about a successful ICBM decrease the payoff of the status quo for North Korea equal to or below zero. This represents an equal or lower payoff than being attacked by the United States given North Korea announced a failure (f^F), which certainly is questionable.

Overall, the one key policy implication to take away from this analysis is that what allows the Pyongyang regime to survive is their ability to either provide a credible threat, or at least arouse enough certainty around their capabilities as to discourage a US attack. The more difficult North Koreas tests are to evaluate, and more generally, the more difficult their general capabilities are to determine, the more secure the regime is, as the US will presumably only attack when swift victory is assured.

Conclusion

This paper presents an extensive game which shows that North Korea would never signal to the world whether it is capable of deploying an ICBM against the US when there are no costs for North Korea to provide proof. Hence, the model has clear limitations as some assumptions are too strong or too far from real international political dynamics. Altering these assumptions would give rise to further work and hopefully further insight. However, the model does show the value of uncertainty to the North Korean regime. The uncertainty which surrounds their nuclear program and capabilities, and their country in general, is what allows their regime to survive. This perhaps gives insight into why the regime undertakes such erratic behavior, that this behavior preserves uncertainty around their capability to inflict disaster upon the US and its allies. This indicates that there may in fact be some strategic method behind Mr. Kim's apparent madness.

References List:

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3. Kong, K. 2017. 'North Korea Test-Fires ICBM That Could Put Entire U.S. in Range'. Bloomberg. [on-line], https://www.bloomberg.com/news/articles/2017-11-28/north-korea-launches-another-ballistic-missile-yonhap-says. [Accessed: 30 November 2017].
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Appendix

Game 1: US Intelligence community observes a failed DPRK ICBM test

- Probability ICBM development is successful: 0.1
- Probability ICBM development is unsuccessful: 0.9

Strategy Sets

The strategy sets are

$$S_{DPRK} = \{(s^S s^F), (s^S f^F), (f^S s^F), (f^S s^F)\}$$

$$S_{USA} = \{(R^s R^f), (R^s A^f), (A^s R^f), (A^s A^f)\}$$

Bi-Matrix

The game can be rewritten as a bi-matrix in which the numbers are weighted by the underlining probability chosen by nature. The bi-matrix is as follows with best responses underlined:

	R^sR^f	R^sA^f	A^sR^f	A^sA^f
s^Ss^F	$\underline{5}$, 1.1	$\underline{5}$, 1.1	0 , $\underline{4.5}$	$\underline{0}$, $\underline{4.5}$
s^Sf^F	$\underline{5}$, 1.1	0.5 , $\underline{4.7}$	4.5 , 0.9	$\underline{0}$, 4.5
f^Ss^F	$\underline{5}$, 1.1	4.5 , 0.9	0.5 , $\underline{4.7}$	$\underline{0}$, 4.5
f^Sf^F	$\underline{5}$, 1.1	0 , $\underline{4.5}$	$\underline{5}$, 1.1	$\underline{0}$, $\underline{4.5}$

The set of pure-strategy Nash equilibria, and hence (since the game has no proper subgames) the set of pure-strategy subgame perfect Nash equilibria, of the game are:

$$SPNE = \{(s^Ss^F, A^sA^f), (f^Sf^F, A^sA^f)\}$$

We know that any strategy profile which is part of a PBE assessment must be a SPNE, and so we can restrict our attention to the SPE strategy profiles found above.

Range of beliefs that those SPNE can be part of a PBE

Let α denote the probability with which the USA believes the ICBM development was successful when the DPRK announces a success (USA is at far left node),

$$\alpha = \text{Prob}(S|s)$$

Let q denote the probability with which the USA believes the ICBM development was successful when the DPRK announces a failure (USA is a second most furthest left node),

$$q = \text{Prob}(S|f)$$

First: Test SPNE=(s^Ss^F,A^sA^f)

After observing the announcement of success (s), the USA beliefs are:

$$\alpha = Prob(S|s) = \frac{p(s|S) * p(S)}{p(s|S) * p(S) + p(s|F) * p(F)} = \frac{1 * 0.1}{(1 * 0.1) + (1 * 0.9)} = 0.1$$

After observing the off-the-equilibrium message of failure (f), the USA beliefs are:

$$q = Prob(S|f) = \frac{p(f|S) * p(S)}{p(f|S) * p(S) + p(f|F) * p(F)} = \frac{0 * 0.1}{(0 * 0.1) + (0 * 0.9)} = \frac{0}{0}$$

Hence, belief consistency requires that α=0.1 and places no restriction on q (off-the equilibrium beliefs). Hence, we can have any $q \in [0,1]$.

In situations when success is announced (s), we have (using α=0.1) that:

$$U_{USA}(s^S s^F, R^s) = 0.1(2) + 0.9(1) = 1.1$$

$$U_{USA}(s^S s^F, A^s) = 0.1(0) + 0.9(5) = 4.5$$

$$U_{USA}(s^S s^F, R^s) < U_{USA}(s^S s^F, A^s)$$

So, it is optimal for the USA to attack when the DPRK announces a successful ICBM development.

When failure is announced (f), we have

$$U_{USA}\left(s^S s^F, R^f\right) = q(2) + (1-q)(1) = 1 + q$$

$$U_{USA}\left(s^S s^F, A^f\right) = q(0) + (1-q)(5) = 5 - 5q$$

$$U_{USA}\left(s^S s^F, R^f\right) \leq U_{USA}\left(s^S s^F, A^f\right) \text{ if and only if (iff)}$$

$$1 + q \leq 5 - 5q$$

$$6q \leq 4$$

$$q \leq \frac{2}{3}$$

Therefore, $s_{USA} = A^s A^f$ is only sequentially rational for the USA when $q \in [0, {}^2/_3]$. We need to check that $s_{DPRK} = s^S s^F$ is sequentially rational for the DPRK in this scenario. This is very straight forward, since $s_{USA} = A^s A^f$ implies that the USA will always attack. So, it doesn't matter what the DPRK does (they will receive a payoff of 0 irrespectively), making all strategies sequentially rational including $s_{DPRK} = s^S s^F$.

$$PBE = \left\{\left(s^S s^F, A^s A^f\right), \alpha = 0.1, q) \; with \; q \in \left[0, \frac{2}{3}\right]\right\}$$

Second: Test SPNE=($f^S f^F$, $A^s A^f$)

Belief consistency places no restrictions on α (off-the equilibrium beliefs- so we can have any $\alpha \in [0,1]$) and requires that q=0.1.

In situations when failure is announced (f), we have (using q=0.1) that

$$U_{USA}\left(f^S f^F, R^f\right) = 0.1(2) + 0.9(1) = 1.1$$

$$U_{USA}\left(f^S f^F, A^f\right) = 0.1(0) + 0.9(5) = 4.5$$

$$U_{USA}\left(f^S f^F, R^f\right) < U_{USA}\left(f^S f^F, A^f\right)$$

So, it is optimal for the USA to attack when the DPRK announces an unsuccessful ICBM development (f).

When success is announced (s), we have

$$U_{USA}(f^S f^F, R^s) = a(2) + (1-a)(1) = 1 + a$$

$$U_{USA}(f^S f^F, A^s) = a(0) + (1-a)(5) = 5 - 5a$$

$$U_{USA}(f^S f^F, A^s) \leq U_{USA}(f^S f^F, R^s) \text{ iff}$$

$$5 - 5a \leq 1 + a$$

$$a \geq \frac{2}{3}$$

Therefore, $s_{USA} = A^s A^f$ is only sequentially rational for the USA when $\alpha \in [0, {}^2/_3]$. We need to check that s_{DPRK}=\ f^Sf^F is sequentially rational for the DPRK in this scenario. This is very straight forward, since s_{USA}=\ A^sA^f implies that the USA will always attack. So, it doesn't matter what the DPRK does (they will receive a payoff of 0 irrespectively), making all strategies sequentially rational including $s_{DPRK} = s^S s^F$.

$$PBE = \left\{ \left(f^S f^F, A^s A^f\right), \alpha, q = 0.1\right) \textit{ with } \alpha \in \left[0, \frac{2}{3}\right] \right\}$$

To summarize, the set of PBE is:

$$PBE = \left\{ \left(s^S s^F, A^s A^f\right), \alpha = 0.1, q\right) : \ q \in \left[0, \frac{2}{3}\right] \right\} \cup \left\{ \left(f^S f^F, A^s A^f\right), \alpha, q = 0.1\right) : \ \alpha \in \left[0, \frac{2}{3}\right] \right\}$$

Game 2: US Intelligence community observes a successful DPRK ICBM test

- Probability ICBM development is successful: 0.9
- Probability ICBM development is unsuccessful: 0.1

Strategy Sets

The strategy sets are the same as above.

Bi-matrix

The new game can be rewritten as a bi-matrix in which the numbers are weighted by the changed underlining probability chosen by nature. The bi-matrix is as follows with best responses underlined:

	R^sR^f	R^sA^f	A^sR^f	A^sA^f
s^Ss^F	$\underline{5}$, $\underline{1.9}$	$\underline{5}$, $\underline{1.9}$	0 , 0.5	$\underline{0}$, 0.5
s^Sf^F	$\underline{5}$, 1.9	4.5 , $\underline{2.3}$	0.5 , 0.1	$\underline{0}$, 0.5
f^Ss^F	$\underline{5}$, 1.9	0.5 , 0.1	4.5 , $\underline{2.3}$	$\underline{0}$, 0.5
f^Sf^F	$\underline{5}$, $\underline{1.9}$	0 , 0.5	$\underline{5}$, $\underline{1.9}$	$\underline{0}$, 0.5

The set of pure-strategy Nash equilibria, and hence (since the game has no proper subgames) the set of pure-strategy subgame perfect Nash equilibria, of the game are

$$SPNE = \{(s^Ss^F, R^sR^f), (s^Ss^F, R^sA^f), (f^Sf^F, R^sR^f), (f^Sf^F, A^sR^f)\}$$

We know that any strategy profile which is part of a PBE assessment must be a SPNE, and so we can restrict our attention to the SPE strategy profiles found above.

Range of believes that those SPNE can be part of a PBE

The notation of the believes of the USA and their meaning are identical to the prior game.

First: Test SPNE=(s^Ss^F, R^sR^f)

Belief consistency requires that α=0.9 and places no restriction on q (off-the equilibrium beliefs). So, we can have any q∈[0,1].

In situations when success is announced (s), we have (using α=0.9) that

$$U_{USA}(s^Ss^F, R^s) = 0.9(2) + 0.1(1) = 1.9$$

$$U_{USA}(s^Ss^F, A^s) = 0.9(0) + 0.1(5) = 0.5$$

$$U_{USA}(s^Ss^F, A^s) < U_{USA}(s^Ss^F, R^s)$$

So, it is optimal for the USA to retreat when the DPRK announces a successful ICBM test.

When failure is announced (f), we have

$$U_{USA}(s^Ss^F, R^f) = q(2) + (1-q)(1) = 1 + q$$

$$U_{USA}(s^Ss^F, A^f) = q(0) + (1-q)(5) = 5 - 5q$$

$$U_{USA}(s^Ss^F, A^f) \leq U_{USA}(s^Ss^F, R^f) \text{ iff}$$

$$5 - 5q \leq 1 + q$$

$$q \geq \frac{2}{3}$$

Therefore, $s_{USA} = R^sR^f$ is only sequentially rational for the USA when $q \in [^2/_3, 1]$.

We need to check that $s_{DPRK} = s^Ss^F$ is sequentially rational for the DPRK in this scenario. This is very straight forward, since $s_{USA} = R^sR^f$ implies that the USA will always retreat so it doesn't matter what the DPRK does (they will receive a payoff of 5 irrespectively), making all strategies sequentially rational including $s_{DPRK} = s^Ss^F$.

$$PBE = \left\{\left(s^Ss^F, R^sR^f\right), \alpha = 0.9, q\right) \text{ with } q \in \left[\frac{2}{3}, 1\right]\right\}$$

Second: Test SPNE=(s^Ss^F, R^sA^f)

Belief consistency requires that α=0.9 and places no restriction on q (off-the equilibrium beliefs); so we can have any $q \in [0,1]$.

In situations when success is announced (s), we have (using α=0.9) proven above that

$$U_{USA}(s^Ss^F, A^s) < U_{USA}(s^Ss^F, R^s)$$

So it is optimal for the USA to retreat when the DPRK announces a successful ICBM development.

When failure is announced (f), we have

$$U_{USA}\left(s^Ss^F, R^f\right) \leq U_{USA}\left(s^Ss^F, A^f\right) \text{ iff}$$

$$1 + q \leq 5 - 5q$$

$$q \leq \frac{2}{3}$$

Therefore, $s_{USA} = R^sA^f$ is only sequentially rational for the USA when $q \in [0, ^2/_3]$.

We need to check that $s_{DPRK} = s^Ss^F$ is sequentially rational for the DPRK in this scenario. This is very straight forward, since $s_{DPRK} = s^Ss^F$ yields DPRK the highest utility with respect to $s_{USA} = R^sA^f$(U_{DPRK}=5)

$$U_{DPRK}\left(s^Ss^F, R^sA^f\right) > U_{DPRK}\left(s^Ss^{F'}, R^sA^f\right)$$

This makes $s_{DPRK} = s^Ss^F$ sequentially rational for DPRK.

$$PBE = \left\{\left(s^Ss^F, R^sA^f\right), \alpha = 0.9, q\right) \text{ with } q \in \left[0, \frac{2}{3}\right]\right\}$$

Third: Test SPNE=(f^Sf^F,R^sR^f)

Belief consistency places no restrictions on α (off-the equilibrium beliefs- so we

can have any $\alpha \in [0,1]$) and requires that q=0.9.
In situations when failure is announced (f), we have (using q=0.9) that

$$U_{USA}(f^S f^F, R^f) = 0.9(2) + 0.1(1) = 1.9$$

$$U_{USA}(f^S f^F, A^f) = 0.9(0) + 0.1(5) = 0.5$$

$$U_{USA}(f^S f^F, A^f) < U_{USA}(f^S f^F, R^f)$$

So, it is optimal for the USA to retreat when the DPRK announces an unsuccessful ICBM test (f).
When success is announced (s), we have

$$U_{USA}(f^S f^F, R^s) = a(2) + (1-a)(1) = 1 + a$$

$$U_{USA}(f^S f^F, A^s) = a(0) + (1-a)(5) = 5 - 5a$$

$$U_{USA}(f^S f^F, A^s) \leq U_{USA}(f^S f^F, R^s) \text{ iff}$$

$$5 - 5a \leq 1 + a$$

$$a \geq \frac{2}{3}$$

Therefore, $s_{USA} = R^s R^f$ is only sequentially rational for the USA when $\alpha \in [^2/_3, 1]$. We need to check that $s_{DPRK} = f^S f^F$ is sequentially rational for the DPRK in this scenario. This is very straight forward, since $s_{USA} = R^s R^f$ implies that the USA will always retreat so it doesn't matter what the DPRK does (they will receive a payoff of 5 irrespectively), making all strategies sequentially rational including $s_{DPRK} = f^S f^F$.

$$PBE = \left\{ \left(f^S f^F, R^s R^f\right), \alpha, q = 0.9) \text{ with } \alpha \in \left[\frac{2}{3}, 1\right] \right\}$$

Fourth: Test SPNE=($f^S f^F$, $A^s R^f$)

Belief consistency places no restrictions on α (off-the equilibrium beliefs- so we can have any $\alpha \in [0,1]$) and requires that q=0.9.
In situations when failure is announced (f), we have (using q=0.9) proven above that

$$U_{USA}(f^S f^F, A^f) < U_{USA}(f^S f^F, R^f)$$

So, it is optimal for the USA to retreat when the DPRK announces an unsuccessful ICBM test (f).
When success is announced (s), we have using the results obtained in step three

$$U_{USA}(f^S f^F, R^s) \leq U_{USA}(f^S f^F, A^s) \text{ iff}$$

$$1 + a \leq 5 - 5a$$

$$\alpha \leq \frac{2}{3}$$

Therefore, $s_{USA} = A^s R^f$ is only sequentially rational for the USA when $\alpha \in [0, {}^2/_3]$. We need to check that $s_{DPRK} = f^S f^F$ is sequentially rational for the DPRK in this scenario. This is very straight forward, since $s_{DPRK} = f^S f^F$ yields DPRK the highest utility with respect to $s_{USA} = A^s R^f$ ($U_{DPRK} = 5$).

$$U_{DPRK}\left(f^S f^F, A^s R^f\right) > U_{DPRK}\left(f^S f^{F'}, A^s R^f\right)$$

This makes $s_{DPRK} = f^S f^F$ sequentially rational for DPRK.

$$PBE = \left\{\left(f^S f^F, A^s R^f\right), \alpha, q = 0.9\right) \text{ with } \alpha \in \left[0, \frac{2}{3}\right]\right\}$$

To summarize, the set of PBE is:

$$PBE = \left\{\left(s^S s^F, R^s R^f\right), \alpha = 0.9, q\right) : \; q \in \left[\frac{2}{3}, 1\right]\right\} \cup$$
$$\left\{\left(s^S s^F, R^s A^f\right), \alpha = 0.9, q\right) : \; q \in \left[0, \frac{2}{3}\right]\right\} \cup$$
$$\left\{\left(f^S f^F, R^s R^f\right), \alpha, q = 0.9\right) : \; \alpha \in \left[\frac{2}{3}, 1\right]\right\} \cup$$
$$\left\{\left(f^S f^F, A^s R^f\right), \alpha, q = 0.9\right) : \; \alpha \in \left[0, \frac{2}{3}\right]\right\}$$